**The relationship between FT, DTFT, DFT and Z-Transform**

* **Fourier Transform (FT):**

The **Fourier transform** is named after Joseph Fourier.

**Definition**:

  Mathematical transformation employed to transform signals between time (or spatial) domain and frequency domain, the new function is then known as the **Fourier transform** and/or the frequency Spectrum of the function *f.*

Mathematically it is defined as

F (ω) =   (1)

Where

 ω = 2f  f = physical frequency in eq. 1

 F (ω) = the signal in frequency domain

 F (t) = is the signal in time domain.

The Fourier transform is, in general, a complex function. We can express it as the sum of its real and imaginary components, or in exponential form, that is, as

F(w) Re{F(w)} + jIm{F(w)} F(w) ejφw

The *Inverse Fourier transform* is defined as

f(t) =

* **DTFT: Discrete-Time Fourier Transform:**

Now simply look at Eq. 1. The right hand side is a continuous integral. This is impossible in engineering since the signals we can get are series (discrete) sampled periodically.

If we evaluate the z-transform at , then we get the DTFT -- this evaluation is equivalent to evaluating the z-transform on the unit circle in the complex plane.

The DTFT is a special case of the z-transform .Fourier Transform performed on a time domain sequence is the DTFT. The expressions for the DTFT *X(ejω)* and the IDTFT *x(n)* are

  (2)

The DTFT and its inverse (IDTFT) are extensively used for the analysis and design of discrete-time systems and in applications of digital signal processing such as speech processing, speech synthesis, and image processing. Remember that the terms *frequency response of a discrete-time signal* and the *discrete-timeFourier transform* (DTFT) are synonymous and will be used interchangeably. This is also known as the *frequency spectrum*; its magnitude response and phase response are generally known as the *magnitude spectrum* and *phase spectrum*, respectively.

* **Discrete Fourier Transform (DFT)**

Definition:

Signal Processing written by Sophocles J. Orfanidis provides a definition to DFT: "The N-point DFT of a length-L signal is defined to be the DTFT evaluated at N equally spaced frequencies over the full NY Quist interval"

In Eq. 2, the FT is discretized in time domain. Computationally, you cannot represent all frequencies since they are uncountable and endless real numbers. So we also need to discretize the FT in frequency domain, the DFT:

 X (ω) k =   (3)

Where (ω) k = 2 π k

N and k are nature numbers between 0 and N-1.

Eq. 3 is called the "N-point DFT of a length-L signal."

* **Discussion: DTFT vs. DFT**
* DTFT is a "time sampling" Fourier Transform and DFT is a "frequency sampling" DTFT. So, DFT is a "time sampling and frequency sampling" Fourier Transform.
* We define DTFT because we have to sample the continual infinite signal whereas we define DFT coz we can't calculate at every frequency point. DTFT discretize the time domain while DFT discretize the frequency domain additionally.
* N only concerns DFT and limits computational frequency resolution. But L concerns both the DTFT and DFT, and limits the physical frequency resolution.
* The D in DTFT emphases the discretization in time domain while the D in DFT emphases the discretization in frequency domain. Indeed, the frequency domain here we mention is digital frequency domain varies from 0 to 2 Pi
* **Z-transform**

The Fourier Transform dos not converge for a numbers of signals but the Laplace Transform does. The Laplace transform allow us to perform transform analysis of unstable systems. Z-transform is the discrete time counter part of the Laplace Transform.

In mathematics and signal processing, the **Z-transform** converts a time domain signal, which is a sequence of real or complex numbers, into a complex frequency domain representation.

It can be considered as a discrete-time equivalent of the Laplace transform. This similarity is explored in the theory of time scale calculus.

Mathematically: the z-transform is given by the sum



Where z is a complex variable. X[z] is the z-transform of the input discrete time signal x[n].

There are a number of important relationships between the z-transform and the Fourier transform. To explore these relationships, let us express the complex variable z in the polar form as

With r as the magnitude of z and Ω. In terms of r and Ω the above equation becomes

From the above equation we see that is the Fourier transform of the sequence x[n] multiplied by a real exponential. The exponential weighting r-n may be decaying or growing with increasing n, depending on whether r is greater than or less than unity. We note that for r = 1, the z-transform reduces to the Fourier transform. Thus, the z-transform reduces to the Fourier transform on the contour in the complex z-plane corresponding to a circle with a radius of unity as shown in figure below.



This circle is called the unit circle. For convergence of the z-transform we require that the Fourier transform of x[n]r-n converge.

